A canonical reduced form for singular time invariant linear systems. Part II

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Abstract:- We consider quadruples of matrices (E, A, B, C), representing singular linear time invariant systems in the form

$$\left.\begin{array}{rl} E\dot{x}(t) & = Ax + Bu \\ y & = Cx \end{array}\right\} \tag{1}$$

with $E, A \in M_{p\times n}(C)$, $B \in M_{p\times m}(C)$ and $C \in M_{q\times n}(C)$under proportional and derivative feedback and proportional and derivative output injection.

In this paper we present a canonical reduced form preserving the structure of the system and we obtain a collection of invariants that they permit us to deduce the canonical reduced form.

Key-Words:- Singular linear systems,proportional and derivative feedback, proportional and derivative output injection, canonical reduced form, structural invariants.

AMS Classification: 15A04, 15A21, 93B52.

1 Introduction

We denote by $M_{p\times n}(C)$ the space of complex matrices having p rows and n columns, and in the case which $p = n$ we write $M_n(C)$.

We consider the set $\mathcal{M}$ of quadruples of matrices (E, A, B, C) representing families of singular linear time invariant systems in the form $E\dot{x} = Ax + Bu$, $y = Cx$ with $E, A \in M_{p\times n}(C)$, $B \in M_{p\times m}(C)$ and $C \in M_{q\times n}(C)$, under equivalence relation that accepting one or more, of the following transformations: basis change in the state space, input space, output space, feedback, derivative feedback, output injection, derivative output injection and premultiplication by an invertible matrix.

The case where one or both of matrices B and C does not appear in the systems, has been largely studied, so we consider the cases where $\operatorname{rank} B > 0$ and $\operatorname{rank} C > 0$.

In [4], a canonical reduced form is presented, but we want to emphasize about the structure of the system, so we present a new canonical reduced form, where the system decompose into independent systems a maximal regular system and a minimal strictly singular system. Also, in this paper, a complete system of invariants given the canonical reduced form is obtained.

In the sequel we will use the following notations.
- I_n denotes the n-order identity matrix,
- N denotes a nilpotent matrix in its reduced form $N = \operatorname{diag}(N_1, \ldots, N_\ell)$, $N_i =$

$\begin{pmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{pmatrix} \in M_{n_i}(C)$,

- J denotes the Jordan matrix $J = \mathrm{diag}(J_1, \dots, J_t)$, $J_i = \mathrm{diag}(J_{i_1}, \dots, J_{i_s})$, $J_{i_j} = \lambda_i I + N$,
- $L = \mathrm{diag} = (L_1, \dots, L_q)$, $L_j = \begin{pmatrix} I_{n_j} & 0 \end{pmatrix} \in M_{n_j \times (n_j+1)}(C)$,
- $R = \mathrm{diag}(R_1, \dots, R_p)$, $R_{n_j} = \begin{pmatrix} 0 & I_{n_j} \end{pmatrix} \in M_{n_j \times (n_j+1)}(C)$
- t, r_c y r_o determine the quantity of controllable and observable, controllable no observable and observable no controllable blocks that appear in an standard triple of matrices.

2 Equivalence of singular systems

We consider singular linear as in (1), many interesting and useful equivalence relations between singular systems have been defined. We deal with the equivalence relation accepting one or more, of the following transformations: basis change in the state space, input space, output space, operations of state and derivative feedback, state and derivative output injection and to pre-multiply the first equation in (1) by an invertible matrix. That is to say.

Definition 1 *Two quadruples $(E_i, A_i, B_i, C_i) \in \mathcal{M}$, $i = 1, 2$, are equivalent if and only if there exist matrices $P \in Gl(n; \mathbb{C})$, $Q \in Gl(p; \mathbb{C})$, $R \in Gl(m; \mathbb{C})$, $S \in Gl(q; \mathbb{C})$, $F_E^B, F_A^B \in M_{m\times n}(\mathbb{C})$, $F_E^C, F_A^C \in M_{p\times q}(\mathbb{C})$ such that*

$$\begin{aligned} E_2 &= QE_1P + QB_1F_E^B + F_EC_1P, \\ A_2 &= QA_1P + QB_1F_A^B + F_AC_1P, \\ B_2 &= QB_1R, \\ C_2 &= SC_1P, \end{aligned}$$

Following [4], we can reduce any quadruple to a simpler form.

Theorem 1 *Let $(E, A, B, C) \in \mathcal{M}$ be a quadruple of matrices. Then it is equivalent under equivalence relation considered, to a quadruple $(E_\omega, A_\omega, I_\omega, I_\omega)$ in the following form:*

$$\left(\begin{pmatrix} \overline{E} & \\ & 0 \end{pmatrix}, \begin{pmatrix} \overline{A} & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & \\ & I_b \end{pmatrix}, \begin{pmatrix} 0 & \\ & I_c \end{pmatrix} \right)$$

where $(\overline{E}, \overline{A})$ is a pair in its Kronecker reduced form.

A collection of structural invariants to the quadruple $(E_\omega, A_\omega, I_\omega, I_\omega)$ characterizing equivalent quadruples is the following collection of numbers

i) $\omega_1 \geq \omega_2 \geq \dots \geq \omega_z \geq 1$ nilpotency indices

ii) $k_1(\lambda) \geq k_2(\lambda) \geq \dots \geq k_{j(\lambda)}(\lambda) \geq 1$ Segre characteristic corresponding to eigenvalue λ

iii) $\epsilon_1 \geq \dots \geq \epsilon_{r_\epsilon} > \epsilon_{r_\epsilon+1} = \dots = \epsilon_{r_k} = 0$ column minimal indices

iv) $\eta_1 \geq \dots \eta_{l_\eta} > \eta_{l_\eta+1} = \dots = \eta_{l_k} = 0$ row minimal indices

3 New canonical reduced form

In [4], a canonical reduced form preserving partition in matrices E, A, B, C is obtained, now we present a new canonical reduced form preserving the structure of the system and providing a decomposition of the system into two independent subsystem, one of them a maximal regular subsystem, the second one a minimal strictly singular subsystem, and each one is given in its canonical reduced form.

Theorem 2 *Let $(E, A, B, C) \in \mathcal{M}$ be a quadruple of matrices. Then it can be reduced under equivalence relation considered, to the following reduced form (E_c, A_c, B_c, C_c):*

$$\left(\begin{pmatrix} I_1 & 0 \\ 0 & E_k \end{pmatrix}, \begin{pmatrix} A_e & \\ & A_k \end{pmatrix}, \begin{pmatrix} B_e & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} C_e & 0 \\ 0 & 0 \end{pmatrix} \right)$$

where (A_e, B_e, C_e) and (E_k, A_k) are in its Kronecker reduced form (see [6], [3] respectively).

Proof.

Let (E, A, B, C) be a quadruple and $(E_\omega, A_\omega, I_\omega, I_\omega)$ its reduced form given in Theorem 1. So, the quadruple $(E_\omega, A_\omega, I_\omega, I_\omega)$ is partitioned in the following manner

$$\left(\begin{pmatrix} E_r & & \\ & E_s & \\ & & 0 \end{pmatrix}, \begin{pmatrix} A_r & & \\ & A_s & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_b & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & I_c \\ 0 & 0 & 0 \end{pmatrix}\right)$$

with

$$(E_r, A_r) = \left(\begin{pmatrix} I_1 & \\ & N \end{pmatrix}, \begin{pmatrix} J & \\ & I_2 \end{pmatrix}\right)$$

and

$$(E_s, A_s) = \left(\begin{pmatrix} L & \\ & L^t \end{pmatrix}, \begin{pmatrix} R & \\ & R^t \end{pmatrix}\right)$$

We proceed by performing the following steps

Step 1: We consider the subquadruple $(E', A'.B'.C')$:

$$\left(\begin{pmatrix} L & & \\ & L^T & \\ & & 0 \end{pmatrix}, \begin{pmatrix} R & & \\ & R^T & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ I_b \end{pmatrix}, \begin{pmatrix} 0 & 0 & I_c \end{pmatrix}\right)$$

and we take the subtriple:

$$\left(\begin{pmatrix} L \\ 0 \end{pmatrix}, \begin{pmatrix} R \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ I_b \end{pmatrix}\right)$$

and we distinguish two cases depending on relation between r_k and b.

a) $r_k \leq b$

In this case the triple is written

$$\left(\begin{pmatrix} L & \\ 0 & \\ & 0 \end{pmatrix}, \begin{pmatrix} R & \\ 0 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ I_{r_c} & 0 \\ 0 & I_t \end{pmatrix}\right),$$

with $r_c = r_k$, $t = b - r_c \geq 0$. Then, it is controllable.

Detailing the subtriple $\left(\begin{pmatrix} L \\ 0 \end{pmatrix}, \begin{pmatrix} R \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ I_{r_c} \end{pmatrix}\right)$, we have

$$\begin{pmatrix} L \\ 0 \end{pmatrix} = \begin{pmatrix} L_1 & & & & 0 \\ 0 & & & & 0 \\ & L_2 & & & 0 \\ & 0 & & & 0 \\ & & \ddots & & \vdots \\ & & & L_{r_\epsilon} & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ I_{r_c} \end{pmatrix} = \begin{pmatrix} 0 & & & & 0 \\ 1 & & & & 0 \\ & 0 & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 0 & 0 \\ & & & 1 & 0 \\ 0 & 0 & \dots & 0 & I_{r_c - r_\epsilon} \end{pmatrix},$$

$$\begin{pmatrix} R \\ 0 \end{pmatrix} = \begin{pmatrix} R_1 & & & & 0 \\ 0 & & & & 0 \\ & R_2 & & & 0 \\ & 0 & & & 0 \\ & & \ddots & & \vdots \\ & & & R_{r_\epsilon} & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}.$$

It is easy to verify that the controllability indices of each subsystem

$$\left(\begin{pmatrix} L_i \\ 0 \end{pmatrix}, \begin{pmatrix} R_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_i \end{pmatrix}\right) = \left(\begin{pmatrix} I_{\epsilon_i} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_{\epsilon_i} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$i = 1, \dots, r_c$ are $k_i^{c\bar{o}} = \epsilon_i + 1$, corresponding to the subsystem

$$\left(\begin{pmatrix} I_{n_{c\bar{o}}} & \\ & 0 \end{pmatrix}, \begin{pmatrix} A_{c\bar{o}} & \\ & 0 \end{pmatrix}, \begin{pmatrix} B_{c\bar{o}} & \\ & I_t \end{pmatrix}\right).$$

Fixed t, we have that the maximal quantity of observable blocks in the pair (L^T, R^T) is $r_o = c - t$. Then if $l_k > r_o$, we have that the subtriple $\left(\begin{pmatrix} L^T & 0 \end{pmatrix}, \begin{pmatrix} R^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_c \end{pmatrix}\right)$ is completely no observable. We write this triple in the form

$$\left(\begin{pmatrix} L_1^T & 0 & \\ & & L_2^T & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} R_1^T & 0 & \\ & & R_2^T & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_{r_o} & 0 & 0 \\ 0 & 0 & 0 & I_t \end{pmatrix}\right),$$

where the pair (L_1^T, R_1^T) contains the first r_o-blocks and the pair (L_2^T, R_2^T) contains the rest of $l = l_k - r_o$-blocks.

The subtriple $\left(\begin{pmatrix} L_1^T & 0 \end{pmatrix}, \begin{pmatrix} R_1^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_{r_o} \end{pmatrix}\right)$, written in the form

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$$\begin{pmatrix} \begin{pmatrix} L_1^T & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & I_{r_o} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} L_{11}^T\ 0 & & & \\ & L_{12}^T\ 0 & & \\ & & \ddots & \\ & & & L_{1r_\eta}^T\ 0 \\ & & & 0 \end{pmatrix} \\ \begin{pmatrix} 0\ 1 & & & & \\ & 0\ 1 & & & \\ & & \ddots & & \\ & & & 0\ 1 & \\ 0\ 0\ 0\ 0\ \dots & & & 0\ 0 & I_{r_o - l_\eta} \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} R_1^T & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & I_{r_o} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} R_{11}^T\ 0 & & & \\ & R_{12}^T\ 0 & & \\ & & \ddots & \\ & & & R_{1r_\eta}^T\ 0 \\ & & & 0 \end{pmatrix} \\ \begin{pmatrix} 0\ 1 & & & & \\ & 0\ 1 & & & \\ & & \ddots & & \\ & & & 0\ 1 & \\ 0\ 0\ 0\ 0\ \dots & & & 0\ 0 & I_{r_o - l_\eta} \end{pmatrix} \end{pmatrix},$$

is observable and observability indices of each subsystem

$$\left(\begin{pmatrix} L_{1i}^T & 0 \end{pmatrix}, \begin{pmatrix} R_{1i}^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & C_{1i} \end{pmatrix}\right) =$$

$$\left(\begin{pmatrix} I_{\eta_i} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ I_{\eta_i} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \end{pmatrix}\right)$$

$i = 1, \dots, r_o$ are $k_i^{\bar{c}o} = \eta_i + 1$, corresponding to the subsystem

$$(I_{n_{\bar{c}o}}, A_{\bar{c}o}, C_{\bar{c}o}).$$

We observe that, if $r_k \leq b$, then the pencil $\begin{pmatrix} \lambda E_k + A_k & I_b \\ I_c & 0 \end{pmatrix}$ has column full rank. It will have row full rank if and only if $l_k = r_o$, that is to say, $l_k \leq c$.

b) $r_k > b$

In this case, if $l_k > c$, the subquadruple (E', A', B', C') is no controllable and no observable and it can be decomposed into two independent subtriples: a triple no controllable

$$\left(\begin{pmatrix} L_1 & & \\ 0 & & \\ & & L_2 \end{pmatrix}, \begin{pmatrix} R_1 & & \\ 0 & & \\ & & R_2 \end{pmatrix}, \begin{pmatrix} 0 \\ I_b \\ 0 \end{pmatrix}\right),$$

where the pair (L_1, R_1) contains the first $r_c = b$-blocks and the pair (L_2, R_2) contains the rest $r = r_k - b$-blocks and the other no observable triple

$$\left(\begin{pmatrix} L_1^T & 0 & \\ & & L_2^T \end{pmatrix}, \begin{pmatrix} R_1^T & 0 & \\ & & R_2^T \end{pmatrix}, \begin{pmatrix} 0 & I_c & 0 \end{pmatrix}\right),$$

where the pair (L_1^T, R_1^T) contains the firts $r_o = c$-blocks and (L_2^T, R_2^T) contains the rest of the $l = l_k - c$-blocks. Obviously, $t = 0$.

Now, we take the subtriple

$$\left(\begin{pmatrix} L^T & 0 \end{pmatrix}, \begin{pmatrix} R^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_c \end{pmatrix}\right)$$

whose study is analogous to the previous one. If $l_k \leq c$, then the pencil $\begin{pmatrix} \lambda E_k + A_k & I_b \\ I_c & 0 \end{pmatrix}$ has row full rank. It will have column full rank if and only if $r_k = b - t$, that is to say, $r_k \leq b$, where $t = c - l_k$.

Step 2: Now we consider the regular subquadruple (E'', A'', B'', C'') obtained in step 1. If $t \neq 0$, the subtriple (E'', B'', C'') of this subquadruple in the form

$$\left(\begin{pmatrix} I_1 & & & \\ & I_{n(\lambda)} & & \\ & & I_2 & \\ & & & 0 \\ & & & N \end{pmatrix}, \begin{pmatrix} B_{c\bar{o}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I_t \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & C_{\bar{c}o} & 0 & 0 \\ 0 & 0 & 0 & I_t & 0 \end{pmatrix}\right).$$

We separate the subquadruple (E_1, A_1, B_1, C_1):

$$\left(\begin{pmatrix} 0 & \\ & N \end{pmatrix}, \begin{pmatrix} 0 & \\ & I_3 \end{pmatrix}, \begin{pmatrix} I_t \\ 0 \end{pmatrix}, \begin{pmatrix} I_t & 0 \end{pmatrix}\right)$$

the nilpotent matrix N with z blocks.

a) $z \leq t$

Detailing the subquadruple (E_1, A_1, B_1, C_1)

$$E_1 = \begin{pmatrix} 0 & & & & & \\ & N_1 & & & & \\ & & \ddots & & & \\ & & & 0 & & \\ & & & & N_z & \\ & & & & & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & & & & & \\ & I_{31} & & & & \\ & & \ddots & & & \\ & & & 0 & & \\ & & & & I_{3z} & \\ & & & & & 0 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & I_{t-s} \end{pmatrix},$$

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and

$$C_1 = \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & I_{t-s} \end{pmatrix}.$$

Is is easy to verify that each subsystem $(E_{1i}, A_{1i}, B_{1i}, C_{1i})$, $i = 1, \dots, z$ in the form

$$E_{1i} = \begin{pmatrix} 0 & & \\ & & 0 \\ & I_{\omega_i - 1} & \end{pmatrix}, \; A_{1i} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & I_{\omega_i - 1} \end{pmatrix}$$

$$B_{1i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; C_{1i} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},$$

is controllable and observable and the controllable-observable indices are $k_i^{co} = \omega_i + 1$.

b) $z > t$

In this case, the subquadruple (E_1, A_1, B_1, C_1) can be decomposed as

$$\left(\begin{pmatrix} 0 & & \\ & N_1 & \\ & & N_2 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & I_{31} & \\ & & I_{32} \end{pmatrix}, \begin{pmatrix} I_t \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} I_t & 0 & 0 \end{pmatrix} \right)$$

where (N_1, I_{31}) contains the first t blocks of nilpotency and (N_2, I_{32}) the rest $s = z - t$-blocks.

□

4 Structural invariants

Let $(E, A, B, C) \in \mathcal{M}$ be a quadruple and all numbers t, r_c, r_o, and

i) Controllability no observable indices and column minimal indices

$$\begin{aligned} k_i^{c\bar{o}} &= k_i^{\epsilon}, \quad i = 1, \dots, r_c \\ \epsilon_j &= k_{r_c+j}^{\epsilon} - 1, \quad j = 1, \dots, r = r_k - r_c \end{aligned}$$

ii) Observability no controllable indices and row minimal indices

$$\begin{aligned} k_i^{\bar{c}o} &= k_i^{\eta}, \quad i = 1, \dots, r_o \\ \eta_j &= k_{r_o+j}^{\eta} - 1, \quad j = 1, \dots, l = l_k - r_o \end{aligned}$$

iii) Controllable and observable indices and nilpotency indices

a) $t \geq z$

$$\begin{aligned} k_i^{co} &= k_i^{\omega} + 1, \quad i = 1, \dots, z \\ k_i^{co} &= 1, \quad i = z + 1, \dots, t \end{aligned},$$

b) $t < z$

$$\begin{aligned} k_i^{co} &= k_i^{\omega} + 1, \quad i = 1, \dots, t \\ \omega_j &= k_{t+j}^{\omega}, \quad j = 1, \dots, s = z - t \end{aligned}$$

iii) Segre characteristic corresponding to the eigenvalue λ

$$(k_1(\lambda), k_2(\lambda), \dots, k_{j_0(\lambda)}(\lambda))$$

correspond with the structural invariants of the quadruple (E_c, A_c, B_c, C_c).

The procedure presented in §3 gives a simple form to obtain the structural invariants under equivalence relation considered that permit us to obtain a canonical reduced form for a system preserving the system structure.

Theorem 3 *In the set $\mathcal{M}$ of quadruples of matrices (E, A, B, C) under equivalence relation considered, the following collection of numbers*

i) $(r_1^{\mathcal{CO}} \geq r_2^{\mathcal{CO}} \geq \cdots \geq r_{\ell_1}^{\mathcal{CO}} \geq r_{\ell_1+1}^{\mathcal{CO}} = \cdots = 0)$

ii) $(r_0^{\mathcal{C}} \geq 0; r_0^{\mathcal{C}\overline{\mathcal{O}}} \geq r_1^{\mathcal{C}\overline{\mathcal{O}}} \geq \cdots \geq r_{\ell_2-1}^{\mathcal{C}\overline{\mathcal{O}}} \geq r_{\ell_2}^{\mathcal{C}\overline{\mathcal{O}}} = \cdots = 0)$

iii) $(r_0^{\mathcal{O}} \geq 0; r_0^{\overline{\mathcal{C}}\mathcal{O}} \geq r_1^{\overline{\mathcal{C}}\mathcal{O}} \geq \cdots \geq r_{\ell_3-1}^{\overline{\mathcal{C}}\mathcal{O}} \geq r_{\ell_3}^{\overline{\mathcal{C}}\mathcal{O}} = \cdots = 0)$

iv) $(r_1^{\overline{\mathcal{CO}}}(\lambda) \geq r_2^{\overline{\mathcal{CO}}}(\lambda) \geq \cdots \geq r_{\ell(\lambda)}^{\overline{\mathcal{CO}}}(\lambda) \geq r_{\ell(\lambda)+1}^{\overline{\mathcal{CO}}}(\lambda) = \cdots = 0), \quad \lambda \in \mathbb{C}$

constitutes a complete system of invariants.

Proof.

The non-zero r-numbers permit us to deduce the collection of numbers

i) $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_z \geq 1$

ii) $k_1(\lambda) \geq k_2(\lambda) \geq \cdots \geq k_{j(\lambda)}(\lambda) \geq 1, \quad \lambda \in \sigma(E, A, B, C)$

iii) $\epsilon_1 \geq \cdots \geq \epsilon_{r_{k_\epsilon}} > \epsilon_{r_{k_\epsilon}+1} = \cdots = \epsilon_{r_k} = 0$

iv) $\eta_1 \geq \dots \eta_{l_{k_\eta}} > \eta_{l_{k_\eta}+1} = \cdots = \eta_{l_k} = 0$

v) $b \geq 0$

vi) $c \geq 0$

whose correspond with the structural invariants of the quadruple (E_k, A_k, I_b, I_c) and the collection

i) $k_1^{co} \geq k_2^{co} \geq \cdots \geq k_t^{co} \geq 1$

ii) $k_1^{c\bar{o}} \geq k_2^{c\bar{o}} \geq \cdots \geq k_{r_c}^{c\bar{o}} \geq 1$

iii) $k_1^{\bar{c}o} \geq k_2^{\bar{c}o} \geq \cdots \geq k_{r_o}^{\bar{c}o} \geq 1$

iv) $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_s \geq 1$

v) $k_1(\lambda) \geq k_2(\lambda) \geq \cdots \geq k_{j(\lambda)}(\lambda) \geq 1, \quad \lambda \in \sigma(E, A, B, C)$

vi) $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_{r_\epsilon} > \epsilon_{r_\epsilon+1} = \cdots = \epsilon_r = 0$

vii) $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_{l_\eta} > \eta_{l_\eta+1} = \cdots = \eta_l = 0$

whose correspond with the structural invariants of the quadruple (E_c, A_c, B_c, C_c). $\square$

5 Conclusion

In this paper we present a canonical reduced form that gives us a partition of the system into independent subsystems i) completely controllable and observable, ii) completely controllable no observable, iii) completely observable no controllable, iv) containing only all zeros of the system, v) containing only all non transferable zeros and vi) a minimal completely singular subsystem. We obtain a complete system of invariants characterizing the equivalence classes and providing this decomposition where each one of the subsystem is in its canonical reduced form.

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